Twenty-first International Olympiad, 1979

1979/1. Let p and q be natural numbers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that p is divisible by 1979.

1979/2. A prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as top and bottom faces is given. Each side of the two pentagons and each of the line-segments A_iB_j for all $i, j = 1, ..., 5$, is colored either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Show that all 10 sides of the top and bottom faces are the same color.

1979/3. Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal.

1979/4. Given a plane π, a point P in this plane and a point Q not in π, find all points R in π such that the ratio $(QP + PA)/QR$ is a maximum.

1979/5. Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying the relations

$$\sum_{k=1}^{5} kx_k = a, \sum_{k=1}^{5} k^3 x_k = a^2, \sum_{k=1}^{5} k^5 x_k = a^3.$$

1979/6. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there.. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$,

$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}), n = 1, 2, 3, \cdots,$$

where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$.

Note. A path of n jumps is a sequence of vertices $(P_0, ..., P_n)$ such that

(i) $P_0 = A, P_n = E$;

(ii) for every $i, 0 \leq i \leq n-1, P_i$ is distinct from E;

(iii) for every $i, 0 \leq i \leq n-1, P_i$ and P_{i+1} are adjacent.